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A Survey of 1961 Russian Results

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INVARIANT SUBSPACES OF LINEAR TRANSFORMATIONS IN HILBERT SPACE, A SURVEY OF 1961 RUSSIAN RESULTS

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If T is a bounded linear transformation of a Hilbert space $\mathcal{H}$ into itself, two fundamental problems are (1) to determine the invariant subspaces for T, and (2) to use them to reconstruct the transformation. If T has finite dimensional range, the spectrum of T consists of isolated points and contour integration gives a complete answer to both problems. If T is a completely continuous transformation, it can be approximated in the operator norm by transformations of finite dimensional range and there is a similar pattern for invariant subspaces. The origin is the only possible point of accumulation in the spectrum of T. The non-zero spectrum corresponds to eigenvalues for T or its adjoint and so determines invariant subspaces. But the existence of invariant subspaces is not obvious if the origin is the only point in the spectrum of T.

Such transformations are called Volterra transformations because of their close relationship to integral transforms. The existence of invariant subspaces for Volterra transformations is due to von Neumann. The first proof was published by Aronzajn and Smith in 1954. A Volterra transformation T admits invariant subspaces $\mathcal{M}$ in which the restriction of T has arbitrarily small trace norm. If P is the projection of $\mathcal{H}$ into $\mathcal{M}$, then invariance is equivalent to the algebraic condition $PTP = TP$. We may choose P so that $\tau(TP)$ is arbitrarily small. It is possible to choose a family $(P(t))$ of such projections with the following properties: the parameter t

ranges in some closed set of non-negative numbers, which are called regular numbers, as opposed to singular numbers t for which $P(t)$ is not defined; the parametrization is increasing in the sense that $P(s)\,P(t) = P(s)$ whenever $s \le t$ are regular; the parametrization is continuous in the sense that $T\,P(t)$ depends continuously on t in the trace norm; if (s, t) is any interval of singular points with regular end points, then $P(s)TP(t) = TP(t)$; furthermore we may have $P(0) = 0$, and either $P(c) = 1$ for some regular number c if T is trace finite, or $\lim \tau(P(t)) = \infty$ as $t \to \infty$. Whenever such a family of projections is chosen, then for every regular $a > 0$,

$$TP(a) = \int_0^a P(t)(T-T^*)dP(t)$$

with convergence of Stieltjes sums in the trace norm. As a result of this integral representation, we get the estimate

$$\tau(T)^2 \le \frac{1}{2}\,\tau(T-T^*)^2$$

provided that T is a Volterra transformation and that $T-T^*$ is of trace class. The theorems are due to Sahnovič in 1959. The stated formulation is that of Gohberg and Krein in the same year.

These results may be summarized broadly by saying that a Volterra transformation T is determined by a knowledge of $T-T^*$ and of a totally ordered family of invariant subspaces. Actually a completely satisfactory theorem exists only when $T-T^*$ is of trace class, but in 1961 Macaev announced more general results

which relax the hypotheses on T-T* . This aspect of Volterra transformations may therefore be considered well in control even though a small area for more work is left.

For Volterra transformations the major unsolved problem is to determine all the invariant subspaces from a knowledge of any single one-parameter family of invariant subspaces. Otherwise stated the problem is to determine the relationship between any two different integral representations of the transformation. A complete answer to this question is known only in the case that T-T* has two dimensional range and does not have two eigenvalues in the same half-plane $y > 0$ or $y < 0$. This comes from my own unpublished work on Hilbert spaces of entire functions and requires a delicate estimate of growth for entire functions of minimal type which are real on the real axis. So far as I know the Russians have no results of this sort.

Completely continuous transformations are a special case of transformations T such that T-T* is completely continuous, and the larger class of transformations is frequently easier to work with for giving proofs. They were introduced by Livschitz in 1954, and much of the theory has been developed by Brodskiĭ.

If a transformation T has the origin as the only point in its spectrum and if T-T* is completely continuous, then T is necessarily completely continuous, and it is of trace class if T-T* is of trace class. But in general, if T-T* is completely continuous, the spectrum of T may contain more than one point. However, all non-real points in the spectrum are isolated. If

(λ_n) is an enumeration of the non-real spectrum of T, we must have

$$\sum |\lambda_n - \bar{\lambda}_n| \leq \tau(T-T^*)$$

provided that $T-T^*$ is of trace class. Non-real points in the spectrum of T are eigenvalues for T or their conjugates are eigenvalues for T^*, and they correspond to invariant subspaces for T.

If T is a linear transformation with real spectrum and if (a, b) is any finite interval, there are two basic problems: (1) Does there exist a non-zero element f of $\mathcal{H}$ such that $(T-z)^{-1}f$ is analytic across (a, b) ? (2) Is the set of all such elements f a closed subspace of $\mathcal{H}$? Both questions are easily given an affirmative answer if suitable estimates are known for the resolvent of T. All such conditions are adapted from Chapter 8 of Levinson, Gap and density theorems, generalization of a problem of Polya. They arise in showing the existence of non-constant entire functions of minimal exponential type bounded on a sequence of real points. More generally, if $W(x) \geq 1$ is a function of real x, does there exist a non-constant entire function $F(z)$ of minimal exponential type such that $|F(x)| \leq W(x)$ for all real x ? A sufficient condition is that $\log W(x)$ be uniformly continuous and that

$$\int (1 + t^2)^{-1} \log W(t)\, dt = \infty .$$

This is one of my unpublished results.

If T is a bounded linear transformation and if $T-T^*$ is completely continuous, the following information is known about invariant subspaces. There exist invariant subspaces $\mathcal{M}$ such that the restriction S of T to $\mathcal{M}$ has $S-S^*$ arbitrarily small in the trace norm. If $T-T^*$ is of trace class and if a is any real number, then T has an invariant subspace $\mathcal{M}$ such that T restricted to $\mathcal{M}$ has its spectrum in the half-plane $x \geq a$ and T restricted to the orthogonal complement of $\mathcal{M}$ has its spectrum in the half-plane $x \leq a$. But it is not known for what transformations T, with $T-T^*$ completely continuous, the same conclusion can be drawn. Very interesting results in this direction were announced in 1961 by Macaev. They are obtained from an estimate of the resolvent of T in terms of the spectrum of $T-T^*$. The estimate is stated without proof and I am unable to derive the result.

Now that the existence of invariant subspaces has been established for some linear transformations in Hilbert space, attention is being focused on the problem of finding all subspaces. The most promising methods are the 1961 results of Brodskiĭ which relate the existence of invariant subspaces to a factorization problem for certain kinds of operator valued analytic functions. The results will be stated in my own interpretation.

Let $\mathcal{C}$ be a given coefficient Hilbert space. By a vector I will always mean an element of this space. If b is a vector, let $\bar{b}$ be the corresponding linear functional on vectors so that

the inner product takes the form $\langle a, b\rangle = \bar{b}a$ for every vector a. By an operator I mean a bounded linear transformation of vectors into vectors. If a and b are vectors let $a\bar{b}$ be the corresponding operator defined by $(a\bar{b})c = a(\bar{b}c)$ for every vector c. The adjoint of an operator A will be written $\bar{A}$.

Let Ω be a region of the complex plane which contains the origin. I will consider Hilbert spaces $\mathcal{H}$ whose elements are vector valued analytic functions $F(z)$ in Ω. I will suppose that for every number w in Ω, the transformation of $\mathcal{H}$ into $\mathcal{C}$ defined by $F(z) \to F(w)$ is continuous. I will suppose also that $R(w)$: $F(z) \to [F(z)-F(w)]/(z-w)$ is a bounded linear transformation of $\mathcal{H}$ into itself for every w in Ω, and that the following identity holds, where I is a fixed operator such that $\bar{I} = -I = I^{-1}$:

$$2\pi\bar{G}(\beta)\, IF(\alpha) = \langle F, R(\beta)G\rangle - \langle R(\alpha)F, G\rangle + (\alpha-\bar{\beta})\langle R(\alpha)F, R(\beta)G\rangle$$

whenever $F(z)$ and $G(z)$ are in $\mathcal{H}$, and α and β are in Ω. Such a space is characterized by an operator valued analytic function $M(z)$ in Ω, which has value 1 at the origin, and which has this property: if

$$K(w,z) = [M(z)\, I\bar{M}(w) - I]/[\pi(z-\bar{w})] ,$$

then $K(w,z)c$ belongs to $\mathcal{H} = \mathcal{H}(M)$ for every vector c and every w in Ω, and

$$\bar{c}F(w) = \langle F(t), K(w,t)c\rangle$$

holds for every $F(z)$ in $\mathcal{H}(M)$. The most general bounded linear transformation in a Hilbert space is unitarily equivalent to $R(0)$ in $\mathcal{H}(M)$ for some choice of $\mathcal{C}$, I , and $M(z)$.

If $\mathcal{H}(M(b))$ is a given Hilbert space of analytic functions in Ω , an invariant subspace for $R(0)$ in $\mathcal{H}(M(b))$ is a Hilbert space $\mathcal{H}(M(a))$ which is contained isometrically in $\mathcal{H}(M(b))$. Therefore the problem of finding invariant subspaces for linear transformations in Hilbert space is equivalent to studying isometric inclusions of Hilbert spaces of analytic functions.

If $\mathcal{H}(M(a))$ is contained isometrically in $\mathcal{H}(M(b))$, then $M(b,z) = M(a,z)M(a,b,z)$ where $M(a,b,z)$ is an operator valued analytic function such that $\mathcal{H}(M(a,b))$ is defined, and $G(z) \longrightarrow M(a,b,z)G(z)$ is a linear isometric transformation of $\mathcal{H}(M(a,b))$ onto the orthogonal complement of $\mathcal{H}(M(a))$ in $\mathcal{H}(M(b))$. Conversely, if $M(b,z) = M(a,z)M(a,b,z)$ is a factorization of operator valued analytic functions, there is an isometric inclusion of $\mathcal{H}(M(a))$ in $\mathcal{H}(M(b))$ provided that some additional conditions are satisfied. Therefore the inclusion problem for such Hilbert spaces of analytic functions is equivalent to a factorization problem for certain kinds of operator valued analytic functions.

These functions $M(z)$ are characterized by positive definiteness of $[M(z)I\overline{M}(w) - I]/(z-\overline{w})$ for z and w in Ω . This in turn is equivalent to having $M(z)$ of the form

$$M(z) = [\phi(z)I + 1]/[\phi(z)I - 1]$$

in Ω , where $\phi(z)$ is an operator valued analytic function for $y > 0$ and for $y < 0$, which satisfies the inequality $[\phi(z) - \overline{\phi}(z)]/(z-\overline{z}) \geq 0$. Brodskiĭ understands the relationship between invariant subspaces and factorization, but he has yet to discover the intermediate spaces of analytic functions.

In closing I would like to say that boundedness for T is inessential as observed by Sekanovskiĭ, and this fact is useful in the following applications. Let $W(x) > 0$ be a differentiable function of real x such that $W'(x)/W(x)$ is bounded. Let $\mathcal{H}$ be the Hilbert space of functions $F(x)$ such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt < \infty$$

and let $H = -iD$ where D is differentiation in $\mathcal{H}$. Then $H\text{-}H^*$ is bounded. The problem is to determine the invariant subspaces for H , or for the resolvent transformations if H is unbounded. If

$$\int (1 + t^2)^{-1} \overset{+}{\log} W(t)\, dt < \infty ,$$

then the entire functions of a given type a which belong to $\mathcal{H}$ are a closed subspace $\mathcal{M} = \mathcal{M}(a)$ which is invariant under H , and frequently the restriction T of H to this subspace has the property that $T\text{-}T^*$ is completely continuous. Once complete continuity is established, Russian results in Hilbert

space yield interesting theorems in function theory. I will mention only a theorem of Beurling and Malliavin from the Stanford Conference last summer, that $\mathcal{M}(a)$ contains a non-zero element for every positive a, no matter how small. It is interesting to compare their proof, which depends on potential theory, with the present one which uses purely Hilbert space methods.

REFERENCES

[1] Aronzajn, N. and Smith, K. T. Invariant subspaces of completely continuous operators. Annals of Math., 60, 345-350 (1954).

[2] Brodskiĭ, M. S. and Livschitz, M. S. Spectral analysis of non-selfadjoint operators and intermediate systems. Uspekhi Mat. Nauk, 13, 3-85 (1958). (Russian)

[3] Brodskiĭ, M. S. On the triangular representation of operators with completely continuous imaginary part. Doklady Akad. Nauk SSSR, 133, 1271-1274 (1960). (Russian)

[4] Brodskiĭ, M. S. On the triangular representation of completely continuous operators whose spectrum is a point. Uspekhi Mat. Nauk, 16, 135-141 (1961). (Russian)

[5] Brodskiĭ, M. S. A criterion of unicellularity for Volterra operators. Doklady Akad. Nauk SSSR, 138, 512-514 (1961). (Russian)

[6] Brodskiĭ, M. S. On the multiplicative representation of some entire operator functions. Doklady Akad. Nauk SSSR, 138, 751-754 (1961). (Russian)

[7] Gohberg, U. T. and Krein, M. G. On completely continuous operators whose spectrum consists of the origin. Doklady Akad. Nauk SSSR, 128, 227-230 (1959). (Russian)